General Relativity and Gravitational Waveforms

Deirdre Shoemaker

Center for Relativistic Astrophysics
School of Physics
Georgia Institute of Technology

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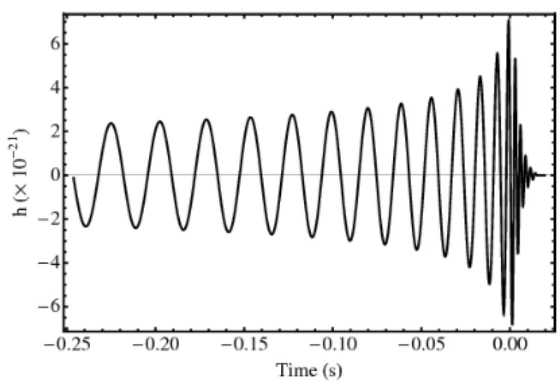
Astrophysics with Gravitational Wave Detections
Copenhagen Niels Bohr Institute

- *Spacetime And Geometry: An Introduction To General Relativity*, Sean Carroll, Pearson (2016), ISBN-10: 9332571651, ISBN-13: 978-9332571655
- *Gravity: An Introduction to Einstein's General Relativity*, James B. Hartle, Pearson (2003), ISBN-10: 0805386629, ISBN-13: 978-0805386622
- *Numerical Relativity: Solving Einstein's Equations on a Computer*. Thomas Baumgarte and Stuart Shapiro, Cambridge University Press, ISBN: 9780521514071
- *Introduction to 3+1 Numerical Relativity*. Miguel Alcubierre, Oxford University Press, ISBN 13:9780199205677
- *Relativistic Hydrodynamics*. Luciano Rezzolla, Oxford University Press, ISBN: 978-0-19-852890-6
- *Astro-GR Online Course on GWs* http://astro-gr.org/online-course-gravitational-waves/
- *2nd Fudan Winter School on Astrophysics Black Holes* Pablo Laguna's and DS's Courses http://bambi2017.fudan.edu.cn/bh2017/Program.html

By the end of these three lectures, I intend for you to

- understand the connection between the gravitational waveform seen in the figure to Einstein's General Theory of Relativity,
- recognize the techniques employed to predict theoretical gravitational waveforms, and what the best use practices are for each,
- and develop some intuition on how the waveform depends on the physical parameters of the black holes.



Lecture 1: General Relativity

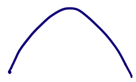
Genera Relativity:

- Gravitational waves: solutions of linearized GR

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

- Weak field sources: solutions to post-Newtonian where you use the fact that the sources are moving slowly and in a weak gravitational field
- Strong field sources (especially merger of black holes): require the full non-linear GR equations

$$G_{\mu\nu} = 0$$



stand on ground and throw eraser
we see parabola for freely falling particle

Jump from table while throwing eraser
we see a straight line

we now have an local inertial frame

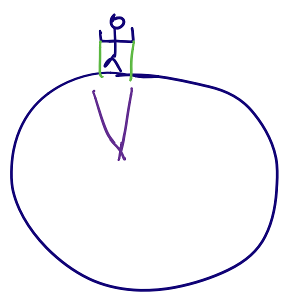
where we can ignore Earth's curvature

x^0, x^1, x^2, x^3 or t, x, y, z

$u^\alpha = \dfrac{dx^\alpha}{d\tau}$ = const. for 0, 1, 2, 3

in LIF u^α = const. for the eraser – a special world line
STRAIGHT 7

Also



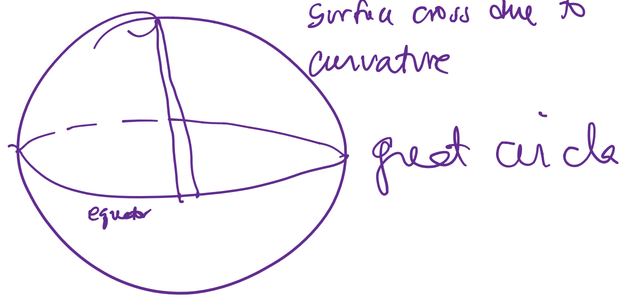
let arms drop due to gravity

Newtonian Physics tells us they will cross

Einstein ?

we know LIF straight lines everywhere along the world line look at geometry

initially $\parallel$ curves on Earth's surface cross due to curvature

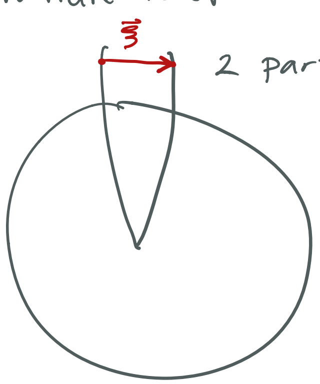


Similarly, Einstein asserts that is what causes crossing of free particle trajectories cross $\Rightarrow$ Newton's gravity is Einstein's curvature

one feature of ST curvature

We need mathematical description of this process, 1st Newtonian and then GR.

Newtonian Perspective:



2 particles, initially // getting drawn together

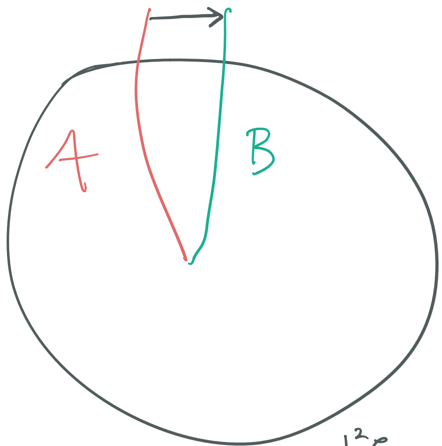
need separation vector $\vec{\xi}$ (3space vector)

$\frac{d\vec{\xi}}{dt} = 0$ because trajectories are initially parallel

$$\frac{d^2\vec{\xi}}{dt^2} = -\underset{\approx}{\mathcal{E}}(\_, \vec{\xi})$$ defn of a tensor

tidal gravity tensor

Derive formula for tidal tensor ...



(A) eqn. of motion $\frac{d^2 x_j}{dt^2} = -\left(\frac{\partial \Phi}{\partial x^j}\right)_A$

– gradient of grav. pot.

(B) $\left(\frac{d^2 x_j}{dt^2}\right)_B = -\left(\frac{\partial \Phi}{\partial x^j}\right)_B$

$\vec{\xi} = (x_j)_A - (x_j)_B$

$$\frac{d^2 \xi_j}{dt^2} = -\underbrace{\left(\frac{\partial^2 \Phi}{\partial x_k \partial x_j}\right)}_{\mathcal{E}} \xi_k$$

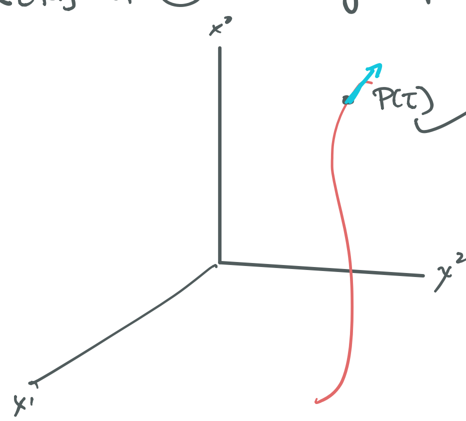
eqn. of tidal gravity in Newtonian potential

components of $\mathcal{E}_{jk}$

$$\mathcal{E}_{jk} = \frac{\partial^2 \Phi}{\partial x_j \partial x_k}$$

TIDAL GRAVITY in GENERAL RELATIVITY

start w/ trajectory of (A) as a single particle



ideal clock; proper time

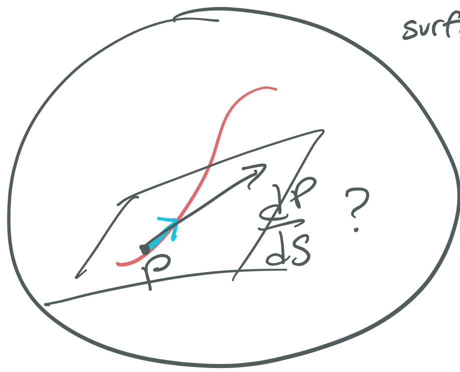
or $x^\alpha(\tau)$

α = 0,1,2,3

tangent vector

$$\mathbf{u} = \frac{d\mathbf{P}}{d\tau} = \lim_{\Delta\tau \to 0} \frac{P(\tau+\Delta\tau) - P(\tau)}{\Delta\tau}$$

what does this look like on surface of Earth?



if we aren't careful, our vectors laid down on surface of a sphere won't make sense

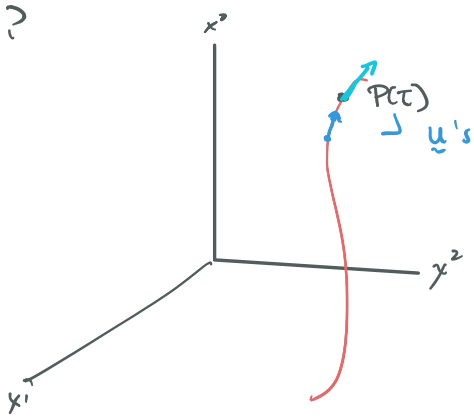
$\frac{dP}{ds}$ lives in tangent space

(linear)

$\therefore \underset{\sim}{u}$ lives in a tangent space. in S.R. it does not matter but in curved ST it does matter!

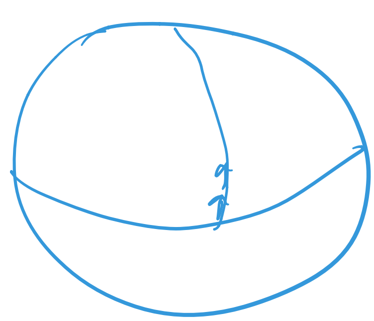
We want to describe worldline of freely falling particle — straight line through curved ST (analog of great circle on surface of Earth)

Straight?



$\underset{\sim}{u}$'s must be $\parallel$ for line to be straight!

not true here



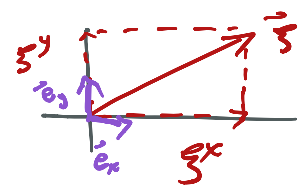
if we project these vectors down onto surface, they will be $\parallel$

little LIF, these will have $\parallel$ components

single LIF @ 1^{st} order is possible

tensor is a linear function of vectors and has some # of slots. If you fill all the slots with vectors, you will get a real # that is linear in vectors you put in slots.

If you leave a slot empty you get out a vector (rather than $\mathbb{R}$)



$\vec{\xi} = \sum_j \xi^j e_j$ in Newtonian (up/down does not matter)

$\underset{\sim}{\mathcal{E}}(\vec{e}_j, \vec{e}_k) = \epsilon_{jk}$

$\left[\underset{\sim}{\mathcal{E}}(\vec{e}_x, \vec{e}_y) = \epsilon_{xy} \right]$

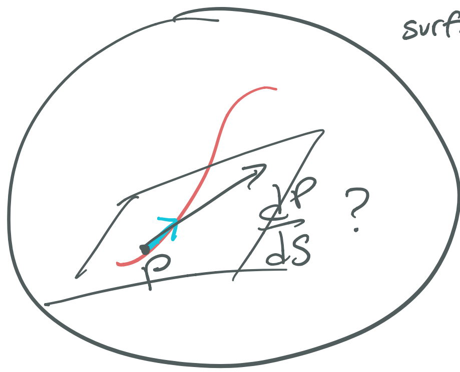
$\underset{\sim}{\mathcal{E}} = \sum_{jk} \epsilon_{jk} \vec{e}_j \otimes \vec{e}_k$ real value function of vectors

return to eqn. for separation : particles being pushed together

$$\frac{d^2\vec{\xi}}{dt^2} = -\underset{\approx}{\mathcal{E}}(\_, \vec{\xi}); \quad \frac{d^2\xi_j}{dt^2} = -\mathcal{E}_{jk}\xi_k$$

✓ no coords needed! component version in a basis (less general) what does this look like on surface of Earth?

if we aren't careful, our vectors laid down on surface of a sphere won't make sense



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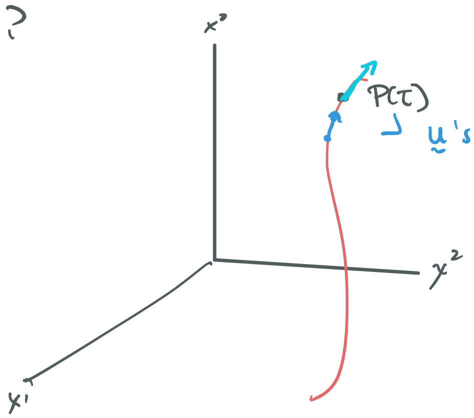

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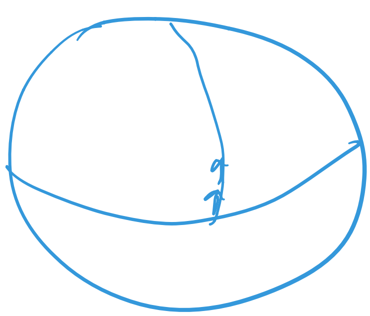

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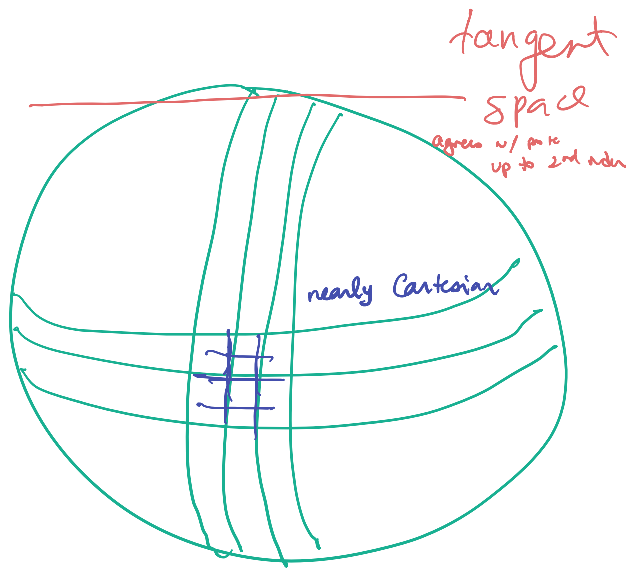
Differentiation of tensor fields

surface of Earth

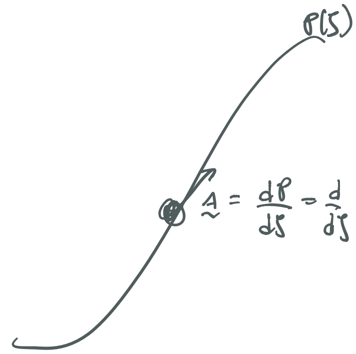


deviations from coordinates being Cartesian occur at 2^{nd} order in distance from origin

$$g_{jk} = \delta_{jk} + O\left(\frac{|\underset{\sim}{x}|^2}{R^2}\right)$$

curvature of Earth is 2^{nd} order

$\Rightarrow$ 1^{st} derivatives can't see curvature



a definition of differentiation

$\mathbb{T}(\_,\_)$ & $\mathbb{T}(P)$

$$\nabla_A \mathbb{T} = \lim_{s\to 0} \frac{\mathbb{T}(P(s+\Delta s))^{\parallel} - \mathbb{T}(P(s))}{\Delta s}$$

these 2 points live on different tangent spaces

almost same $\Delta s \to 0$ but need method to carry the tensor between tangent spaces (to do linear algebra)

Parallel Transport must be same as flat to 1^{st} order in flat ST: $\parallel$ transport means components of tensor are held fixed in local inertial frame or Cartesian coord. system

$\parallel$ transport: insensitive (with one) insensitive to curvature

$$\nabla_A \mathbb{T} = \lim_{s\to 0} \frac{\mathbb{T}(P(s+\Delta s))^{\parallel} - \mathbb{T}(P(s))}{\Delta s}$$

Gradient

$$\nabla_{\vec{A}} \mathbb{T}(\_,\_) \equiv \underset{\sim}{\nabla} \mathbb{T}(\_,\_,\underset{\sim}{A})$$

last

gradient of $\mathbb{T}$

linear function of vectors w/ one more slot than derivative

derivative along $\underset{\sim}{A}$ of $\mathbb{T}$

$$\underset{\sim}{\nabla}\mathbb{T} = T^{\alpha\beta}{}_{;\underline{\mu}}\ \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\beta \otimes \underset{\sim}{e}^\mu,$$

last!

← gradient

how do I compute the components of gradient?

components

Connection Coefficients

$$\nabla_{\underset{\sim}{e}_\mu} \underset{\sim}{e}_\alpha = \Gamma^\beta{}_{\alpha\underline{\mu}}\ \underset{\sim}{e}_\beta$$

last

Theorem: $\nabla_{\underset{\sim}{e}_\mu} \underset{\sim}{e}^\rho = -\Gamma^\rho{}_{\sigma\mu}\ e^\sigma$

$T^{\alpha\beta}{}_{;\gamma}$

if basis vectors change, components would change + we need to take this into account

$$T^{\alpha}{}_{\beta;\gamma} = \underbrace{T^{\alpha}{}_{\beta,\gamma}}_{\partial_{\underset{\sim}{e}_\gamma} T^\alpha{}_\beta \,=\, \frac{\partial}{\partial x^\gamma} T^\alpha{}_\beta} + \Gamma^{\alpha}{}_{\mu\gamma} T^{\mu}{}_{\beta} - \Gamma^{\mu}{}_{\beta\gamma} T^{\alpha}{}_{\mu}$$

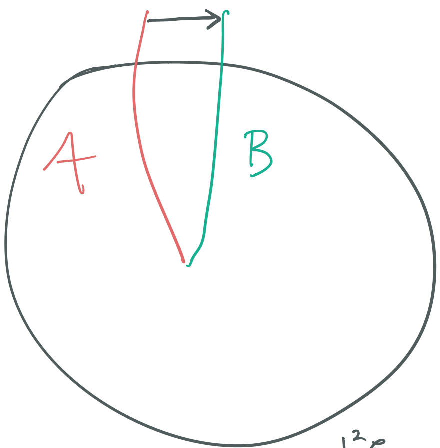
correct

correct

correct lead slot

Derive formula for tidal tensor ...

$$\vec{a} = \frac{\vec{F}}{m} = \vec{g} = -\nabla\Phi$$



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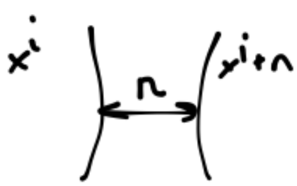
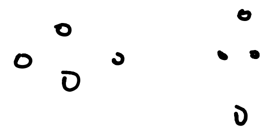
Geodesic Deviation



$$\frac{d^2 \xi^i}{dt^2} \simeq -\eta^{ij} \frac{\Phi}{\partial x^k \partial x^j} n^k$$

This can be used to calculate relative $\vec{a}$ of freely falling



GR $\quad 0 = \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$ & a second geodesic $\bar{x}$

$$0 = \frac{d^2 \bar{x}^\alpha}{d\tau^2} + \bar{\Gamma}^\alpha_{\mu\nu}\frac{d\bar{x}^\mu}{d\tau}\frac{d\bar{x}^\nu}{d\tau}$$

$$\bar{\Gamma}^\alpha_{\mu\nu} \simeq \Gamma^\alpha_{\mu\nu} + n^\sigma\left[\partial_\sigma \Gamma^\alpha_{\mu\nu}\right]$$

$$0 = \frac{d^2(x^\alpha + n^\alpha)}{d\tau^2} + \left[\Gamma^\alpha_{\mu\nu} + n^\sigma \partial_\sigma \Gamma^\alpha_{\mu\nu}\right]\frac{d(x^\mu + n^\mu)}{d\tau}\frac{d(x^\nu + n^\nu)}{d\tau}$$

$$0 = \frac{d^2 n^\alpha}{d\tau^2} + 2\Gamma^\alpha_{\mu\nu} u^\mu \frac{dn^\nu}{d\tau} + n^\sigma\left(\partial_\sigma \Gamma^\alpha_{\mu\nu}\right)u^\mu u^\nu$$

not quite b/c basis vectors

$$\left(\frac{d^2 \underline{n}}{d\tau^2}\right)^\alpha = \partial_\sigma \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\sigma} + \Gamma^\alpha_{\sigma\gamma}\Gamma^\gamma_{\mu\nu} - \Gamma^\alpha_{\gamma\nu}\Gamma^\gamma_{\mu\sigma}\,)\, u^\sigma u^\mu n^\nu$$

$$= \underbrace{-R^\alpha{}_{\mu\nu\sigma}}_{\eta^{ij}\partial_k\partial_j \Phi}\, u^\sigma u^\mu n^\nu$$

Gravity's specialness: dynamical field giving rise to gravitation is the metric tensor describing curvature of ST itself. It is not an additional field propagating through ST.

PRINICIPLE OF EQUIVALENCE: universality of the gravitational interaction

* Weak Equivalence Principle (WEP)
  $m_I = m_G \Rightarrow a = -\nabla\phi$ and freely falling test particles have universal behavior
  $\Rightarrow$ preferred class of trajectories on which unaccelerated particles travel

  Can't distinguish $\vec{a}$ from g for freely falling particles in a sufficiently small region of ST

* Einstein Equivalence Principle (EEP)
  mass not so unique
  In small enough regions of ST, laws of physics reduce to SR. Impossible to detect existence of grav. field by local experiments

  $\Rightarrow$ action of gravity should be attributed to ST curvature
  (no way to shield grav or make a gravitationally neutral object $\therefore$ accel due to grav not uniquely defined)

(+) It is not possible to build a global inertial frame that stretches through ST

(+) Only locally inertial frames are possible

(+) ST is mathematical structure that locally looks like Minkowski but may possess nontrivial curvature over extended regions

Need a mathematical structure: differentiable manifold
— looks locally flat but might have different global geometry
$\mathbb{R}^n$

Can't prove that gravity needs to be thought of as curvature
$\rightarrow$ just do physics

Gravitation

how does gravity influence matter's behavior
how does matter determine gravitational field

$$a = -\nabla \Phi$$
$$\nabla^2 \Phi = 4\pi G \rho \rightarrow \text{matter density}$$

GR: how curvature of ST acts on matter to manifest as gravity
how E-momentum influence ST to create curvature

Galileo: response of matter to gravitation is universal

$$a = -\nabla\phi$$

GR: want a replacement

Exp. 1 $\quad \eta_{\mu\nu} \to g_{\mu\nu}\ ; \quad \partial_\mu \to \nabla_\mu$

$$\underbrace{\frac{d^2x^\mu}{d\lambda^2} = 0}_{\text{not a tensorial eqn.}} = \underbrace{\frac{dx^\nu}{d\lambda}\partial_\nu \frac{dx^\mu}{d\lambda}}_{\text{generalize}} \longrightarrow \frac{dx^\nu}{d\lambda}\nabla_\nu \frac{dx^\mu}{d\lambda}$$

$$\frac{dx^\nu}{d\lambda}\nabla_\nu \frac{dx^\mu}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}$$

free particles move on geodesics

Exp. 2 $\quad \partial_\mu T^{\mu\nu} = 0 \longrightarrow \nabla_\mu T^{\mu\nu} = 0$

But how does this describe gravity?

Einstein Equations

Poisson eqn. $\nabla^2\phi = 4\pi G\rho$

$\nabla^2 = \delta^{ij}\partial_i\partial_j$ Laplacian

$g_{00} \simeq \eta_{00} + h_{00} = 1 - 2\phi$

$\nabla^2 h_{00} = -8\pi G T_{00}$ $\quad [\nabla^2 g]_{\mu\nu} \propto T_{\mu\nu}$ in general

$T_{00} = \rho$

want something like $\Box g_{\mu\nu} = \nabla^\alpha\nabla_\alpha g_{\mu\nu}$ but $\Box g_{\mu\nu} = 0$

$R^\alpha{}_{\mu\nu\sigma}$ is $\partial^2 g$

$R_{\mu\nu} = \kappa T_{\mu\nu}$ no good

$\nabla^\mu T_{\mu\nu} = 0$ conservation of E

$\nabla^\mu R_{\mu\nu} \neq 0$ in general

but the contracted Bianchi Identities

$$\nabla^\mu R_{\mu\nu} - \tfrac{1}{2}\nabla_\nu R = \nabla^\mu\left(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R\right) = \nabla^\mu G_{\mu\nu} = 0$$

$\hookrightarrow \nabla^\mu R_{\mu\nu} = \tfrac{1}{2}\nabla_\nu R$

EINSTEIN EQUATIONS

$$\boxed{G_{\mu\nu} = \kappa T_{\mu\nu}}$$

Newtonian Limit

- particles move slowly w.r.t. light
- gravitational field is weak, namely a perturbation of ST
- gravitational field is static

→ recover N.L. from G.R. using these

Consider the geodesic eqn.

- moving slowly

$$\frac{dx^i}{d\tau} << \frac{dt}{d\tau}$$

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 \qquad \frac{dx^i}{d\tau} \text{ is ignorable}$$

- static gravitational field implies

$$\partial_0 g_{\mu\nu} = 0$$

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\lambda}\left(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}\right)$$

$$= -\frac{1}{2} g^{\mu\lambda} \partial_\lambda g_{00}$$

- Weakness of gravitational field

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad |h_{\mu\nu}| \ll 1$$

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu{}_\sigma$$

$$(\eta^{\mu\nu} + \bar{h}^{\mu\nu})(\eta_{\nu\sigma} + h_{\nu\sigma}) = \delta^\mu{}_\sigma$$

$$\eta^{\mu\nu}\eta_{\nu\sigma} + \eta^{\mu\nu} h_{\nu\sigma} + \bar{h}^{\mu\nu}\eta_{\nu\sigma} + \bar{h}^{\mu\nu} h_{\nu\sigma} = \delta^\mu{}_\sigma$$

$$\delta^\mu_\sigma + \eta^{\mu\nu} h_{\nu\sigma} + \bar{h}^{\mu\nu}\eta_{\nu\sigma} + 0 = \delta^\mu{}_\sigma$$

$$\eta^{\mu\nu} h_{\nu\sigma} + \bar{h}^{\mu\nu}\eta_{\nu\sigma} = 0$$

$$\eta_{\rho\mu}\eta^{\mu\nu} h_{\nu\sigma} = -\bar{h}^{\mu\nu}\eta_{\nu\sigma}\,\eta_{\rho\mu} \qquad -\bar{h}_\rho{}^\nu \eta_{\nu\sigma}$$

$$\delta^\nu_\rho h_{\nu\sigma} = -\bar{h}^{\sigma\nu}\eta_{\mu\nu}\,\eta^{\rho\mu}$$

$$h_{\rho\sigma} = -\delta^\rho_\nu \bar{h}^{\sigma\nu}$$

$$h_{\rho\sigma} = -\bar{h}^{\rho\sigma}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}$$

$$g^{\mu\lambda}\partial_\lambda g_{00}$$

$$(\eta^{\mu\lambda} - h^{\mu\lambda})\,\partial_\lambda(\eta_{00} + h_{00})$$

$$\cancel{\eta^{\mu\lambda}\partial_\lambda\eta_{00}} + \eta^{\mu\lambda}\partial_\lambda h_{00} - \cancel{h^{\mu\lambda}\partial_\lambda h_{00}}$$

$$\therefore \frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\frac{dt}{d\tau}\right)^2 \quad \text{but } \partial_0 h_{00} = 0,$$

$\mu = 0$ component

$$\frac{d^2t}{d\tau^2} = 0$$

$$\left(\frac{d\tau}{dt}\right)^2 \frac{d^2x^i}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \partial_i h_{00}\left(\frac{d\tau}{dt}\right)^2$$

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00} \longrightarrow \text{Newtonian!} \quad h_{00} = -2\Phi\,; \quad g_{00} = -(1+2\Phi)$$

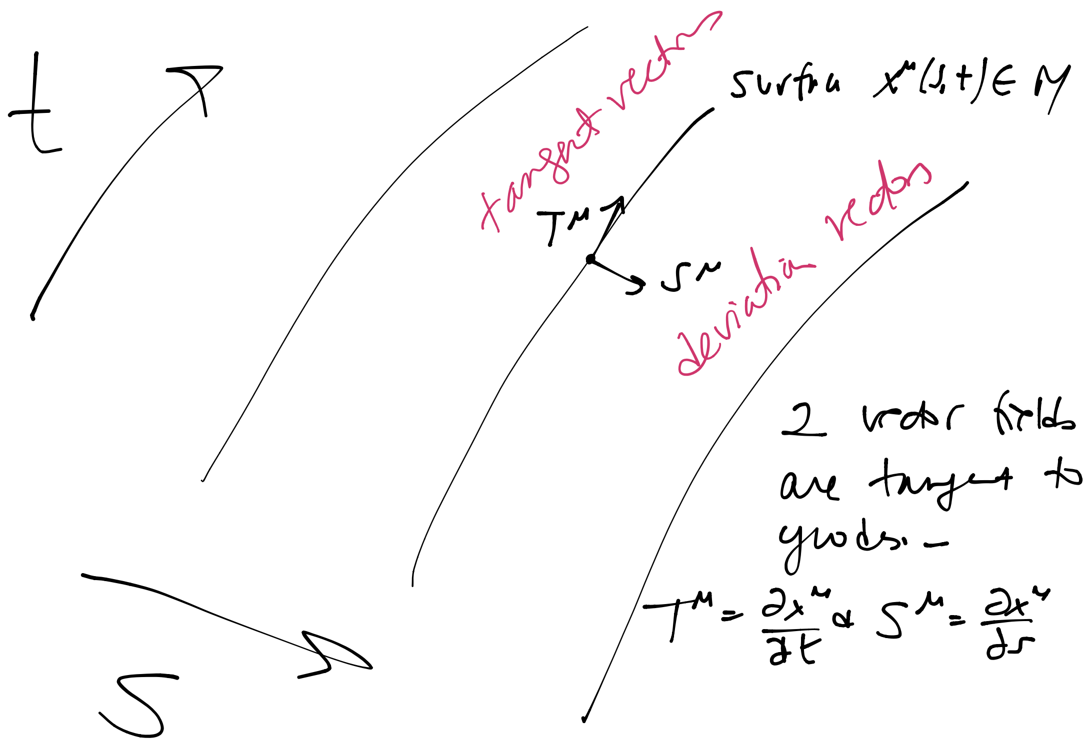
This mean that the curvature of ST is sufficient to describe gravity in our Newtonian limit if $g_{00} = -(1+2\Phi)$

$$a = -\nabla\phi$$

Consider geodesic curve + see how they behave

one parameter family of geodesics $\gamma_s(t)$

$s \in \mathbb{R}$, γ_s is a geodesic parametrized by t



2 vector fields are tangent to geodesics

$T^\mu = \frac{\partial x^\mu}{\partial t}$ & $S^\mu = \frac{\partial x^\mu}{\partial s}$

So S^μ points from 1 geodesic to its neighbor

relative velocity of geodesic

$$V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu$$

relative acceleration of geodesics

$$A^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu$$

S&T are basis vectors $[S,T]=0$ adapted to a coordinate system

$$[S,T]^\mu = S^\lambda \nabla_\lambda T^\mu - T^\lambda \nabla_\lambda S^\mu = 0$$

HW problem $[X,Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu$

So $S^\lambda \nabla_\lambda T^\mu = T^\lambda \nabla_\lambda S^\mu$

$$A^\mu = T^\rho \nabla_\rho V^\mu = T^\rho \nabla_\rho \left(T^\sigma \nabla_\sigma S^\mu \right)$$

$[S, T] = 0$

$$= T^\rho \nabla_\rho S^\sigma \nabla_\sigma T^\mu$$

Leibniz rule

$$= (T^\rho \nabla_\rho S^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$

$\nabla_\rho \nabla_\sigma T^\mu - \nabla_\sigma \nabla_\rho T^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu$ (no torsion)

$$= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu + R^\mu{}_{\nu\rho\sigma} T^\nu)$$

Leibniz rule

$$= (S^\rho \nabla_\rho T^\sigma)(\cancel{\nabla_\sigma} T^\mu) + S^\sigma \nabla_\sigma \underbrace{(T^\rho \nabla_\rho T^\mu)}_{\substack{\text{tangent} \\ \to 0}} - (S^\sigma \nabla_\sigma T^\rho) \cancel{\nabla_\rho} T^\mu + R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

$$= R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

$A^\mu = \frac{D^2}{dt^2} S^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$ Geodesic Deviation

relative a b/t 2 geodesics is $\propto$ to curvature